Surface Tension Handout

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Abstract

Surface tension is a topic that many textbooks neglect. This leaves many olympians confused when given problems about surface tension. This handout attempts to give an introduction to the theory of surface tension as well as giving several examples and problems for the reader to attempt.

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1 Introduction

1.1 The Basics

Surface tension is the tendency of a liquid to shrink into the most minimal surface possible. This phenomenon causes many possibilities: water striders are insects which can walk over water surfaces; water droplets can sit on top of surfaces without seeping through; and through a process called capillary action, plants can move water upwards through its layers. The first question that may come to mind is, why does surface tension want to minimize the shape of water? The reason comes down to the molecular level. If the surface of water is increased, the number of particles on the surface increases, and the molecular energy as well. Increasing energy takes work, so this is why the fluid wants to minimize the energy involved. The incremental energy[1] $\mathrm{d}\mathbf{G}$ of the surface is proportional to its area $\mathrm{d}\mathbf{A}$, so $\mathrm{d}\mathbf{G} = \gamma \mathrm{d}\mathbf{A}$. Here, γ refers to the surface tension (Newtons/meter) of the liquid.

At the surface of a liquid interface, the number of molecular bonds decreases as shown in the image below [4]. This missing energy is essentially positive free energy at the surface, i.e. surface tension. Furthermore, consider the molecules between two interfaces. If there is a molecule A both inside and outside of an interface, there is energy corresponding to that from chemical bonds. Likewise, if there is a molecule B both inside and outside, there is also energy corresponding to that. Accordingly, if there is a molecule A on the outside and molecule B on the inside, there is another energy contribution. In the end, interfaces are the result of different materials on the outside and inside. So, surface tension can be thought of as the energy cost of putting A on the outside and B on the inside. As a result, surface tension always depends on what A and B are, so we must denote it as γ_{AB}. For example, between vapour and liquid, we can denote γ_{VL} or between the solid surface and the liquid, we can denote γ_{SL}.

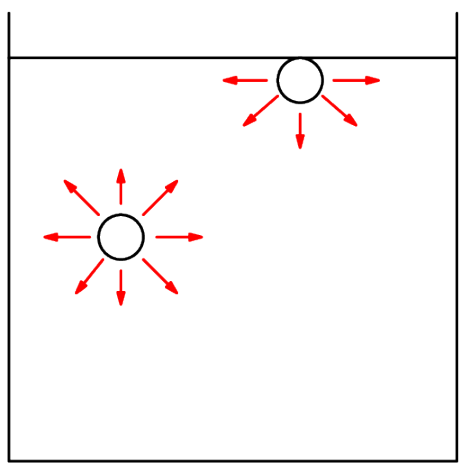


Figure 1: A diagram of the molecular forces of molecules within the liquid. Well within it, molecular bonds act in all directions. However, at the surface, only bonds below the molecule act on it leading to an energy disparity.

We can estimate how large surface tension is. Suppose there is a liquid with density ρ and molar mass M. By ideal gas law, the number of molecules per unit volume is $\rho N_A/M$ where N_A is Avogadro's number. Therefore, at the surface, we can expect the number to go as the 2/3 power. The energy contribution per each molecule is hence, $\gamma/(\rho N_A/M)^{2/3}$. For water, we know from

[1] This is Gibb's free energy.

before that the surface tension at 20° is 72.8 mN/m, which means that putting in realistic numbers gives 0.043 eV per molecule. Compared to the latent heat per molecule, which is approximately 0.45 eV, the effects of surface tension are on the power of 10^{-1}. This quick exercise lets us understand the effects of surface tension more. Surface tension is still relatively small, if not, negligible on larger scales. Another interesting property is that γ changes at different temperatures because the molecular bond energy changes; the relationship is linear with respect to temperature. This was found by the Hungarian physicist Lorand Eötvös and is now known as the Eötvös rule.[2] But, at room temperature some common values for surface temperature are 72.8 mN/m (water), 28.88 mN/m (Benzene), 64.00 mN/m (Glycerol), and 425.41 mN/m (Mercury).

Theorem 1. *For a fluid surface $\mathcal{S}$ with surface tension γ, the infinitesimal force* $\mathrm{d}\mathbf{F}$ *acting on a small length* $\mathrm{d}\mathbf{s}$ *on the surface follows*

$$\mathrm{d}\mathbf{F} = \gamma \mathrm{d}\mathbf{s} \times \hat{\mathbf{n}}. \tag{1}$$

Proof. Suppose we stretch an interface of original length L by a length $\mathrm{d}s$. The change in area is $\mathrm{d}\mathbf{A} = L\mathrm{d}\mathbf{s}$. Therefore, the work done is $\mathrm{d}\mathbf{W} = \gamma L \mathrm{d}\mathbf{s}$. Therefore, the change in force orthogonal to its direction is $\mathrm{d}\mathbf{F} = \gamma \mathrm{d}\mathbf{s}$. The original force that holds the interface is $F = \gamma L$.

Now, we need to make sure that the surface forces are *tangential*. Consider a thin interface of the liquid which stretches from point A to B as shown in . The atmosphere from the air exerts a pressure on the interface while the pressure from the bubble exerts it opposite as well.

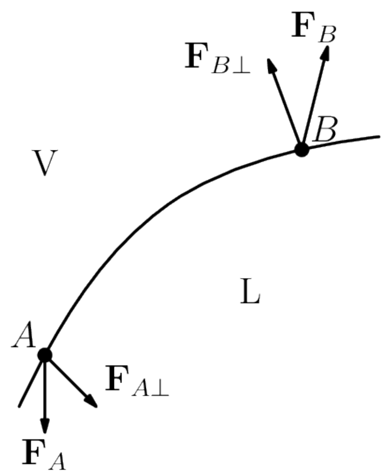


Figure 2: A figure which determines the equilibrium of a vapour liquid interface.

Let $\mathbf{F}_A$ be the force exerted by the liquid at point A and let $\mathbf{F}_B$ be the force exerted by the liquid at point B. We claim that the orthogonal components $\mathbf{F}_{A\perp}$ and $\mathbf{F}_{B\perp}$ must necessarily be zero. Let us assume that AB is small meaning that we can neglect the weight of the interface and assume that pressure is uniform. The net torque from point B will be

$$\boldsymbol{\tau}_B = \mathbf{F}_{A\perp} \cdot AB,$$

and the net torque from point A will be

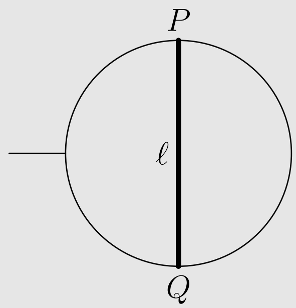
$$\boldsymbol{\tau}_A = \mathbf{F}_{B\perp} \cdot AB.$$

For a rigid body in equilibrium, the torque at any point must be zero. Hence, $\boldsymbol{\tau}_A, \boldsymbol{\tau}_B = 0$ which means $\mathbf{F}_{A\perp}$ and $\mathbf{F}_{B\perp}$ must be zero and that F_A and F_B act along the interface. □

[2]See the appendix for more information.

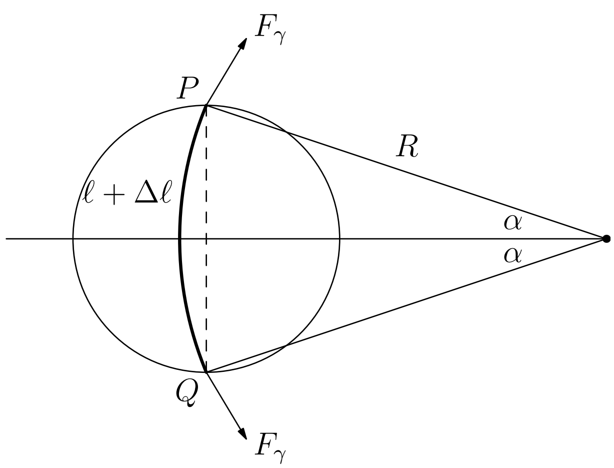
This is similar to how the tension in a rope works. Suppose we cut a part of the interface which has a length L. Then the two ends will pull at each other with a force per unit length γ similar to the tension T in a rope. This fact can be used for many olympiad problems.

Example 1 (KöMaL Magazine) In a wire frame with a diameter of ℓ shown in the figure, a soap membrane with a surface tension of γ was created. The points P and Q are connected by an unstretched hair of length ℓ, young modulus E, and cross section A.



How much does the hair stretch if the membrane is pierced by a hot needle? You can take the approximation $\sin\alpha \approx \alpha - \frac{\alpha^3}{6}$ for small angles α. Assume the weight of the hair is negligible.

Solution. The hot needle serves a purpose of disturbing the membrane forcing it to reach a new equilibrium. The membrane contracts slightly forcing the hair to bend; as the hair is elastic, it stretches.



As the hair bends only slightly, the radius of curvature R of the hair will be large and the angle will be small. The net force of surface tension will be

$$F_{\text{tot}} = 2F_\gamma \sin\frac{\alpha}{2} \approx 2 \cdot \gamma R\alpha \cdot \frac{\alpha}{2} = \gamma R\alpha^2 \approx \gamma\frac{\ell}{2\alpha}.$$

The initial length of the rod is $\ell = 2R\sin\alpha$ and the new length of the rod is $\ell + \Delta\ell = R(2\alpha)$. Hence, $\Delta\ell = 2R(\alpha - \sin\alpha)$. As the rod changes in length, the elastic force is

$$F = \frac{EA\Delta\ell}{\ell} = \frac{2REA(\alpha - \sin\alpha)}{2R\sin\alpha} = \frac{EA(\alpha - \sin\alpha)}{\sin\alpha}.$$

We can approximate the trigonometric expression, using the fact that $\alpha^3 \ll \alpha$ as

$$\frac{\alpha - \sin\alpha}{\sin\alpha} \approx \frac{\alpha - \alpha + \frac{\alpha^3}{6}}{\alpha - \frac{\alpha^3}{6}} \approx \frac{\alpha^3}{6\alpha - \alpha^3} \approx \frac{\alpha^2}{6}.$$

Hence, we can balance forces as

$$\frac{EA\alpha^2}{6} = \gamma\frac{\ell}{2\alpha} \implies \alpha = \sqrt[3]{\frac{3\ell\gamma}{EA}}.$$

From approximations, $\Delta\ell$ can be written as $\frac{\ell\alpha^2}{6}$ meaning

$$\Delta\ell = \frac{\ell}{6}\left(\frac{3\ell\gamma}{EA}\right)^{2/3}.$$

$\square$

1.2 Wetting

When a droplet is placed on a piece of paper it spreads throughout it. On the contrary, when placed on a plastic material, it is stationary. The reason this happens is due to **wetting** which describes the ability of a liquid to maintain contact with a solid surface. Wetting is the result of **adhesive** forces which attempt to spread the liquid across the surface and **cohesive** forces which cause the liquid to ball up. The effect of wetting depends on the contact angle of the fluid, which is the angle the fluid makes at the contact point with the solid. When a droplet is perfectly wetted, it is like a flat pool of water, while on the contrary, if it is non-wetted, it will be exactly a sphere.

Contact Angle	**Type of Wetting**
$\theta = 0°$	Perfect
$0° < \theta < 90°$	High
$90° \leq \theta < 180°$	Low
$\theta = 180°$	Non

Figure 3: A droplet resting on the trichomes of a leaf due to the balance of cohesive and adhesive forces.

Theorem 2. *When a liquid makes contact with a solid surface, it will approach a contact angle that depends on the solid and liquid interfaces as*

$$\gamma_{LV} \cos\theta = \gamma_{SV} - \gamma_{SL}.$$

This is known as **Young's Law** named after Thomas Young [5][6] who made the law in 1805.

Proof. Consider the intersection between the solid, liquid, and vapour interfaces which is known as the triple point. We can create a freebody diagram as shown below. The surface tension between solid and vapour will act outwards from the fluid along their interface line. Similarly, the tension between the liquid and solid will act along their interface line and the tension between liquid and vapour as well.

[INCLUDE IMAGE]

Thus equating the force per unit length along the horizontal axis, we have

$$\gamma_{SV} = \gamma_{SL} + \gamma_{LV} \cos\theta$$

which gives us our equation.

Remark. In the above diagram, there seems to be an unbalanced component $\gamma_{VL} \sin\theta$. Oftentimes, the substrate will have a resisting force that is equal to the vertical force. If the substrate is deformable, it will have a little ridge near the triple point. For the case of olympiad problems, we

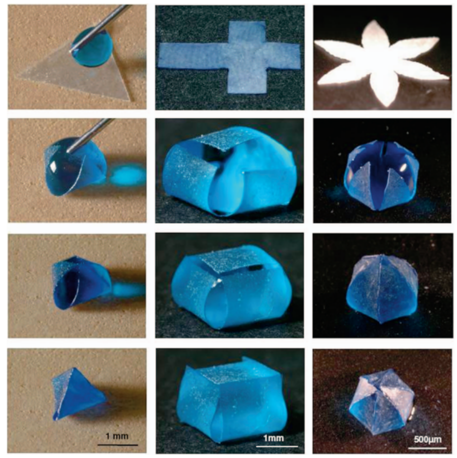


Figure 4: A step-by-step process of how capillary origami happens. These pictures are taken on the scale of $\sim 500\mu$m.

neglect these factors. However, this principle can lead to beautiful structures through a process called capillary origami [2] where scientists cut out a solid medium, put a water droplet on it, and then make it bend into interesting shapes. □

Proof. It is also important to derive this relation with energy, or more specifically, virtual work. This proof is more important to give a full understanding of the importance of Young's equation because we know that surface tension is based on energy, and virtual work can help us solve other problems. So, consider a droplet on the surface as shown below. Let the contact angle change by a small angle $\delta\theta$, and the contact point move a distance δx. This changes the liquid solid area by δA_{LV} which is equal to the change in liquid solid area by

$$\delta A_{LS} = \delta A_{LV} \cos\theta.$$

The total change in energy then follows

$$\begin{aligned}\delta U &= \gamma_{LV}\delta A_{LV} + (\gamma_{SL} - \gamma_{SV})\delta A_{LS} \\ &= (\gamma_{LV}\cos\theta + (\gamma_{LS} - \gamma_{SV}))\delta A_{SL}.\end{aligned}$$

We must have $\delta U/\delta A_{LS} = 0$ for the work to be virtual. Hence,

$$\gamma_{LV}\cos\theta = \gamma_{SV} - \gamma_{SL}.$$

□

Example 2 (Trichomes) The contact angle of liquids is increased in some natural materials due to small hairs called trichomes which reduce the apparent contact area as in figure 3. If the contact angle and area without trichomes is θ and A, then what is the contact angle with trichomes assuming the new contact area is rA where r is a small numerical factor.

Solution. Let the contact angle change by a small angle $\delta\theta$ and the contact point move a distance δx. this changes the liquid solid area by $r\delta A_{LS}$. Furthermore, note that for a fluid to be incompressible, we need $\delta V = 0$ which means that the lost area in the liquid solid area must add up to the liquid vapor area as 1. So,

$$r\delta A_{LS} = (\cos\theta + (1-r))\delta A_{LV}.$$

The total change in energy then follows

$$\begin{aligned}\delta U &= \gamma_{LV}\delta A_{LV} + (\gamma_{SL} - \gamma_{SV})\delta A_{LS} \\ &= (\gamma_{LV}(\cos\theta + (1-r)) + r(\gamma_{LS} - \gamma_{SV}))\delta A_{SL}\end{aligned}$$

We must have $\delta U/\delta A_{LS} = 0$ for the work to be virtual. Hence,

$$r(\gamma_{SV} - \gamma_{LS}) = \gamma_{LV}(\cos\theta + (1-r)).$$

By Young's law, it is easy to replace and find that

$$\cos\theta = r\cos\theta_0 + r - 1.$$

□

Often, we can approximate the droplet as that of a spherical cap as shown in the image below.

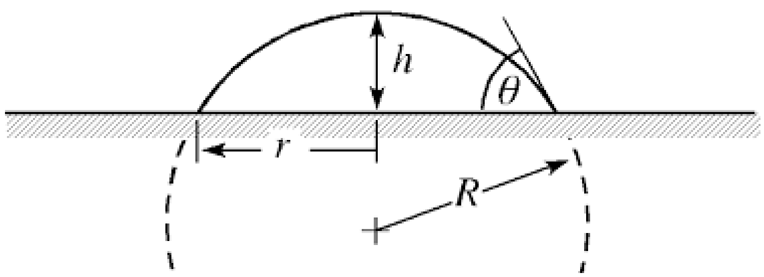


Figure 5: A spherical cap which consists of a sphere that is cut through. It has an inner height h and radius r.

It is well known that the volume of a spherical cap is given as $V = \frac{1}{6}\pi h(3r^2 + h^2)$ and the surface area is $S = \pi(h^2 + r^2)$. The total energy of the droplet will be given as

$$E = \pi r^2(\gamma_{LS} - \gamma_{SV}) + S\gamma_{LV}.$$

We know that under infinitesimal changes, $\mathrm{d}E/\mathrm{d}r = 0$. Hence,

$$\frac{\mathrm{d}E}{\mathrm{d}r} = 0 \implies 2\pi r(\gamma_{LS} - \gamma_{SV}) + \frac{\mathrm{d}S}{\mathrm{d}r}\gamma_{LV}.$$

As S depends on h which depends on r, we cannot immediately differentiate. To combat this, $\mathrm{d}V = 0$ which implies that

$$\mathrm{d}V = 0 \implies \frac{1}{6}\pi(\mathrm{d}h(3r^2 + h^2) + h(6r\mathrm{d}r + 2h\mathrm{d}h)) = 0.$$

This immediately yields the relationship of h upon r as

$$\frac{\mathrm{d}h}{\mathrm{d}r} = -\frac{2rh}{r^2 + h^2}.$$

Furthermore, note that $\mathrm{d}S = 2\pi(h\mathrm{d}h + \mathrm{d}r)$, so

$$\mathrm{d}S = 2\pi\left(-\frac{2rh^2}{r^2+h^2}+1\right)\mathrm{d}r \implies \frac{\mathrm{d}S}{\mathrm{d}r} = 2\pi r\left(\frac{r^2-h^2}{r^2+h^2}\right).$$

Putting this into our equation of energy minimization yields

$$r = h\sqrt{\frac{\gamma_{SV}+\gamma_{LV}-\gamma_{LS}}{\gamma_{LV}+\gamma_{LS}+\gamma_{SV}}}.$$

The actual profile curve of droplets are more complex (taking hyperbolic forms), but the spherical approximation here is useful because many olympiad problems will use spherical or elliptical models.

1.3 Dimensional Analysis and Buckingham π

Oftentimes, surface phenomenon makes it hard to solve problems optimistically. Therefore, a lot of olympiad problems rely on the solver to use dimensional analysis to get reasonable estimates. Dimensional analysis is also a good way to make sure that your answer makes sense.

In short, dimensional analysis is the art of figuring out how some quantity is affected by other parameters in a physical setup. For example, if a raindrop is falling to the ground, there are several different parameters that can determine things such as its terminal velocity: drag coefficient, radius, mass, surface tension, density of surrounding air, etc. In most problems, we have some certain parameters that we are given, neglecting other effects to determine that quantity which is being affected. Consider mathematically, the case of the terminal velocity of the raindrop. Let the terminal velocity be represented by some variable, say A. The other quantities will be combined in some order of operations to have the same dimensions of A, let us call this B. Then, on either side of the equation, we need

$$A = B \implies [A] = [B]$$

where a bracket [] represents the dimensions of that quantity. Dimensions can typically be brought down to **three** fundamental SI units: mass $[M]$, length $[L]$, and time $[T]$. Force, for example, can be written as

$$[F] = [M][a] = [MLT]^{-2}.$$

Consider an example.

Example 3 (2020 $F = ma$ Exam A) Liquid droplets store a given amount of potential energy per surface area due to their surface tension. When two identical, nearly spherical droplets coalesce onto a certain surface, part of this energy can be converted to kinetic energy, causing the coalesced droplet to jump. Assuming the conversion is 100% efficient, how does the maximum height h depend on the radius r of the initial droplets.

Solution. The surface area of the coalesced droplet is $S \propto r^2$. So the total potential energy is $E_p \sim \gamma r^2$. As density is constant, the mass of the droplet is $m \propto r^3$. So, dimensional analysis mandates that both sides of the equation must have the same units, or

$$E_p \sim E_k \implies \gamma r^2 \propto r^3 h \implies h \propto r^{-1}.$$

$\square$

If we have a quantity $[Z]$, it can be written as a combination of multiple exponents from various quantities:

$$[Z] = [A]^{k_1}[B]^{k_2} \dots [Y]^{k_3}.$$

Each quantity $[A]$ will have M, L, and T with some exponents as

$$[M^\alpha L^\beta T^\gamma] = [M^{m_1} L^{\ell_1} T^{t_1}]^{k_1} [M^{m_2} L^{\ell_2} T^{t_2}]^{k_2} \dots [M^{m_n} L^{\ell_n} T^{t_n}]^{k_n}.$$

So, we can have three systems of equations

$$\alpha = \sum_i m_i k_i \tag{2}$$

$$\beta = \sum_i \ell_i k_i \tag{3}$$

$$\gamma = \sum_i t_i k_i \tag{4}$$

This however, raises the concern of what happens if there are more than three variables in our equation. Three systems of equations can solve for three variables, but with four variables, we would need one more equation. This concern is brought up in the Buckingham-π theorem.

Theorem 3. *If there are N quantities with D independent dimensions, then one can form $N-D$ independent dimensionless quantities, but we cannot tell how the answer depends on them.*

In most cases, this will not be needed. But, consider an example (also problem XX).

Example 4 (Falling Droplet) Consider a water droplet as shown below. When it falls on a water hydrophobic surface, a smaller ring of smaller droplets assemble around the water droplet after the collision. Explain why this happens and under what conditions/parameters could we find and measure the dimensions of such a ring?

We could have four different dimensions: the height it is dropped from h, the droplets radius r, its density ρ, and surface tension σ. Buckinghams theorem says that we have 4 quantities, but they have 3 independent dimensions. So there will be 1 dimensionless quantity. Here it is obvious that it has to be the two length parameters. So we will have a dimensionless function $f(r/h)$. Now we can do dimensional analysis.

$$[\sigma] = MT^{-2}$$
$$[\rho] = ML^{-3}$$
$$[h] = L$$

We must then say the dimensions for the radius R to be

$$R = f(r/h)h^{\alpha}\sigma^{\beta}\rho^{\gamma} \implies L^1 = [MT^{-2}]^{\alpha}[ML^{-3}]^{\beta}[L]^{\gamma}.$$

Therefore, we have three systems of equations

$$\alpha + \beta = 0$$
$$-2\alpha = 0$$
$$-3\beta + \gamma = 0$$

Suprisingly, this implies that $\alpha, \beta = 0$ and $\gamma = 1$. So, the new radius is simply $R = f(r/h)h$. This is actually wrong as the radius does depend on surface tension and density, as an exercise, see why so and what variables we neglected. The radius is not necessarily proportional to only h, though. It could also be that $R = f(r/h)r^{139}h^{-137}$. We can only guess the dimensions which is one of the flaws of dimensional analysis when we have too many dimensions.

1.4 Young-Laplace Law

Suppose we have a soap bubble which is a hollow sphere with a thin membrane. This membrane holds a surface tension γ. If we expand its radius by a small amount δr, the energy change δE can be attributed to the change of internal pressure $p\delta V$ and the change in surface energy $\gamma\delta A$. In such a bubble, the total surface area is $8\pi R$ because the soap bubble is **double-layered** due to there being surface tension in the outer and inner layers; hence, twice as much soapy water is created when stretching it by δr. The infinitesimal change in surface tension is hence

$$\delta E_s = \delta(\gamma 8\pi R^2) = 16\pi\gamma R\delta R,$$

while the infinitesimal change in energy due to pressure is

$$\delta E_p = \delta p\delta V = \delta p\delta\left(\frac{4}{3}\pi R^3\right) = \delta p(4\pi R^2)\delta R.$$

Energy is at minimum so it does not change in infinitesimal movement. Hence, $\delta E/\delta R = 0$, meaning

$$\frac{\delta E}{\delta R} = 0 = 16\pi\gamma R + \delta p(4\pi R^2) \implies \delta p = \frac{4\gamma}{R}.$$

If we had a soapy *ball*, we could do the same proof but with the total surface area as $4\pi R^2$ to retrieve $\delta p = 2\gamma/R$. These hold for spherical surfaces, but we can generalize this result in the following theorem. For a cylindrical surface, we can also replicate the same problem to get $\delta p = \gamma/R$ (refer to the appendix for an alternate derivation). We can put all the results into one table.

Shape	**Pressure Difference**
Δp_{bubble}	$4\gamma/R$
Δp_{ball}	$2\gamma/R$
$\Delta p_{\text{cylinder}}$	γ/R

Theorem 4. *For a surface with two principle radii of curvature R_1 and R_2, the change in pressure ΔP can be attributed as*

$$\Delta P = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

The proof is easily accessible online, refer to [3] for instance.

Example 5 (Water between two plates) In a zero gravity region, a drop of liquid assumes a cylindrical shape of diameter D between two plates separated a distance d. If the curved surface of water is at right angles to the plate as shown in the figure, find the force exerted by the drop on the plates.

Solution. Young-Laplace's law says that the two radii of curvature of the droplet are $R_1 = D/2$ and $R_2 \to \infty$. Hence,

$$\Delta p = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{2\gamma}{D}.$$

The force acting on the plate is then

$$F = \Delta p\pi\left(\frac{D}{2}\right)^2 = \frac{2\gamma}{D}\pi\frac{D^2}{4} = \frac{\pi\gamma D}{2}.$$

□

Consider a pool of water. This pool will have some height h due to the interactions of surface tension. If this pool has a density ρ, the pressure from due to height difference from the ground to its maximum height will be $\Delta p = \rho g \Delta z$. This pressure has to be counteracted by the pressure from surface tension, which is conveniently given by the Young Laplace law $\Delta p = \gamma(r_1^{-1} + r_2^{-1})$. One of the radii of curvature will be $r_1 = \infty$ at its maximum height as the pool can be thought to be long, and at a point z, its radius of curvature will be $1/r_2 = \mathrm{d}\theta/\mathrm{d}s$, where θ is the inclination of the surface to the horizontal at that point and s is the distance measured along the arc from the ground. Furthermore, since $\mathrm{d}z/\mathrm{d}s = \sin\theta$, we can write that

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}z} = \frac{1}{r_2 \sin\theta}.$$

Therefore,

$$\rho g z = \gamma(r_1^{-1} + r_2^{-1}) = \gamma \sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}z}.$$

This can be integrated to give

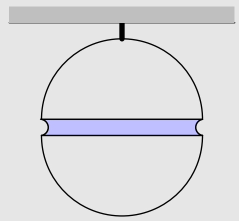
$$\frac{\rho g z^2}{2\gamma} = 1 - \cos\theta \implies h = \sqrt{\frac{2\gamma}{\rho g}(1 - \cos\theta)}.$$

NB! This can also be derived with the principle of virtual work with the tools of wetting seen in the previous subchapter.

This result is very useful and is one of the most well known problems related to surface tension, because it provides a lesson of how ubiquitous the Young Laplace law is.

2 Examples

Example 6 (2021 Online Physics Olympiad Open Round) Anyone who's had an apple may know that pieces of an apple stick together, when picking up one piece a second piece may also come with the first piece. The same idea is tried on a *golden apple*. Consider two uniform hemispheres with radius $r = 4$ cm made of gold of density $\rho_g = 19300$. The top half is nailed to a support and the space between is filled with water.



Given that the surface tension of water is $\gamma = 0.072$ and that the contact angle between gold and water is $\theta = 10^\circ$, what is the maximum distance between the two hemispheres so that the bottom half doesn't fall? Answer in millimeters.

Solution. Let h be the difference in height. There are 3 forces on the bottom hemisphere. The force from gravity, which has magnitude $\frac{2}{3}\pi\rho_g g r^3$, the force from the surface tension, and the force from the pressure difference at the top and bottom. The pressure difference is given by the young-laplace equation,

$$\Delta P = \gamma\left(-\frac{1}{r} + \frac{2\cos\theta}{h}\right) \approx \frac{2\gamma\cos\theta}{h}. \tag{5}$$

The radii are found by some simple geometry. It is likely that r will be much larger than the height, so we can neglect the $1/r$ term. Now the force from surface tension is $2\pi r\gamma\sin\theta$, since we take the vertical component. So we can now set the net force to 0,

$$\frac{2}{3}\pi\rho_g g r^3 = \pi r^2\Delta P + 2\pi r\gamma\sin\theta \implies \frac{2}{3}\rho_g g r^2 = (2r\gamma\cos\theta)\frac{1}{h} + 2\gamma\sin\theta. \tag{6}$$

This is simple to solve,

$$h = \frac{2r\gamma\cos\theta}{2\rho_g g r^2/3 - 2\gamma\sin\theta} = 2.81\times10^{-5}\text{ m} = 0.0281\text{ mm}. \tag{7}$$

This is very small so our approximation from earlier is justified. □

Example 7 (2018 Eötvös Competition) There exists an air bubble with a volume $V = 1\ \text{cm}^3$ at normal pressure in a closed, long, cylindrical container filled with water at room temperature. The container is rotated slowly about its central axis of symmetry at a steady state, carefully accelerating it in a state of weightlessness until it reaches an angular velocity of $= 300\ \text{s}^1$ where it is kept rotating at the same constant speed. What shape does the air bubble now take? Give the typical dimensions of the bubble. The surface tension of water is $\gamma = 0.07$ N/m.

Solution. First, let us assume that the air bubble has a lower density than water. When rotating in the water, there is a pressure gradient that pushes the bubble to the center of the centrifuge. As the centrifugal force is radial, the bubble has to elongate in the axial direction because volume must be conserved. As $\omega \gg 1$, we can assume that the the bubble becomes a very long ellipsoid which is similar to a cylinder with length $L \gg 1$ and radius $r \ll 1$. We can proceed through 2 different methods now.

Method 1 (Energy)

Method 2 (Force) By Young-Laplace law for a cylindrical surface, we have

$$\Delta p = \frac{\gamma}{r} \implies p(r) - p(0) = \frac{\gamma}{r}. \tag{8}$$

The centrifugal force per unit volume varies as $F(r) = \rho\omega^2 r$. This implies that the centrifugal pressure is

$$p(r) = \int \mathbf{F}(r) \cdot \mathrm{d}\mathbf{r} = \frac{1}{2}\rho\omega^2 r^2 + c. \tag{9}$$

where c is a constant. This tells us that our constant is

$$c = p(0) - \frac{\gamma}{r} - \frac{1}{2}\rho\omega^2 r^2. \tag{10}$$

□

Example 8 (Glass Plate) A glass plate is inserted into a fluid surface as shown below such that the fluid makes a contact angle θ with the plate. What is the change in height of the fluid given density ρ and surface tension γ?

Solution. Consider a part of the liquid with a horizontal length L. In the horizontal direction, there will be four forces applied on the fluid.These are the surface tension force that is directed at the top and bottom, the reaction force from the wall N, and the force due to atmospheric pressure F_p. From force balancing, we find that

$$\gamma L + N - F_p - \gamma L \sin\theta = 0.$$

Let us now also consider a point A on the fluid that is a height y from the bottom with pressure P_0. We find the pressure at point B is

$$P_B = P_0 - \rho g y.$$

Integrating this expression (or simply taking the average) gives us

$$\int_0^h P_B = \int_0^h (P_0 - \rho g y) dy \implies N = \left(P_0 - \frac{\rho g h}{2}\right) Lh.$$

The force due to atmospheric pressure is $F_p = P_0 Lh$. Substituting all these values into our equation gives us

$$\gamma L + \left(P_0 - \frac{\rho g h}{2}\right) Lh - P_0 Lh - \gamma L \sin\theta = 0.$$

Canceling factors and solving for h finally gives us

$$\Delta h = \sqrt{\frac{2\gamma}{\rho g}(1 - \sin\theta)}.$$

$\square$

Example 9 (Coalescing Bubbles) Two soap bubbles of diameters a and b coalesce together to form a larger bubble of diameter c. If the surface tension of the solution is γ, find the atmospheric pressure of the environment.

Solution. First, note that the pressure inside the bubble with radius r is larger than the outside pressure because of the Laplace component $4\gamma/r$, where 4 takes into account that the bubble has two surfaces. If the number of moles of gas inside the bubbles are ν_a and ν_b respectively, then from the ideal gas law

$$\left(P_0 + \frac{4\gamma}{a/2}\right) \cdot \frac{4}{3}\pi \left(\frac{a}{2}\right)^2 = \nu_a RT$$

and

$$\left(P_0 + \frac{4\gamma}{b/2}\right) \cdot \frac{4}{3}\pi \left(\frac{b}{2}\right)^2 = \nu_b RT.$$

For the total merged bubble. we have that

$$\left(P_0 + \frac{4\gamma}{c/2}\right) \cdot \frac{4}{3}\pi \left(\frac{c}{2}\right)^2 = (\nu_a + \nu_b) RT$$

where T is the air temperature. Putting all these equations together gives us

$$\left(P_0 + \frac{8\gamma}{a}\right) a^3 + \left(P_0 + \frac{8\gamma}{b}\right) b^3 = \left(P_0 + \frac{8\gamma}{c}\right) c^3.$$

Factoring out both sides gives us

$$\begin{aligned} P_0(a^3 + b^3 - c^3) &= 8\gamma(c^2 - a^2 - b^2) \\ P_0 &= \frac{8\gamma(c^2 - a^2 - b^2)}{a^3 + b^3 - c^2} \end{aligned}$$

□

Example 10 (2014 Seagull Competition) Due to instability (know as the Plateu-Rayleigh instability, a tap water stream breaks into droplets at a certain height. This process can be modelled by the instability of a long water cylinder in weightlessness. Let the diameter of the cylinder be $d = 1$ mm; estimate the time period T during which the amplitude of the most unstable perturbations will increase by a factor of $e \approx 2.718$. The surface-tension of water $\gamma = 72$ g/s^2, and the density $\rho = 1$ g/cm^3.

Solution. Since the potential energy in a sphere is lower, let us assume that the potential energy stored in the surface is going to completely transferred to kinetic energy instantaneously. Let us determine the radius of this sphere. Since the volume is equal, we have:

$$\pi(d/2)^2 h = \frac{4}{3}\pi r^3 \implies r = \frac{1}{2}\left(\frac{3d^2 h}{2}\right)^{1/3}$$

Therefore, the change in energy is:

$$\gamma \Delta S = \gamma \pi d h - \gamma 4\pi \left(\frac{1}{2}\left(\frac{3d^2 h}{2}\right)^{1/3}\right)^2 = \gamma \pi h \sqrt[3]{h}\left(\sqrt[3]{h} - \sqrt[3]{9d/4}\right)$$

Let us assume this change in energy causes half of the liquid to move at a speed of v. Then:

$$\frac{1}{2}(0.5m)v^2 = \gamma \pi h \sqrt[3]{h}\left(\sqrt[3]{h} - \sqrt[3]{9d/4}\right) \implies v = 4\sqrt{\frac{\gamma}{\rho d}}\sqrt{\frac{\sqrt[3]{h} - \sqrt[3]{9d/4}}{\sqrt[3]{h}}}$$

using the fact that $m = \rho\pi(d/2)^2 h$. If we take v to be the average velocity, then the characteristic time would be given by:

$$t = \frac{h}{v} = \frac{1}{4}\sqrt{\frac{\rho d}{\gamma}}\sqrt{\frac{\sqrt[3]{h^7}}{\sqrt[3]{h} - \sqrt[3]{9d/4}}}$$

We see that the characteristic time depends on the height h of the original cylinder. Since we want the time for the most unstable perturbations, we want to maximize t and we can do this by taking the derivative. Doing so gives us:

$$h = \left(\frac{7}{6}\right)^3 \frac{9}{4} d \approx 3.57d$$

Therefore, the characteristic time is:

$$t = \frac{1}{4}\sqrt{\frac{\rho d}{\gamma}}\sqrt{\frac{\sqrt[3]{(3.57d)^7}}{\sqrt[3]{3.57d} - \sqrt[3]{9d/4}}} \approx 0.0088 \text{ s}$$

Solution 2. Dimensional analysis tells us that the characteristic time is:

$$t = \sqrt{\frac{\rho d^3}{\gamma}} = 0.003727 \text{ s}$$

□

3 Olympiad Problems

Problem 1. *2011 IPhO Problem 2: An Electrified Soap Bubble*

Problem 2. *2014 IPhO Problem 1 B: Three Problems*

Problem 3. *2012 IPhO Problem 2: Kelvin Water Dropper*

Problem 4. *2011 APhO Problem 3: Birthday Balloon*

Problem 5. *2010 APhO Problem 3: Electrons and Gas Bubbles in Liquids*

Problem 6. *2008 APhO Problem 2: Tea Ceremony and Physics of Bubbles*

Problem 7. *2016 IZhO: Equilibrium in Terms of Energy*

Problem 8. *2020 USAPhO B1: String Cheese*

Problem 9. *2008 USAPhO: Optical Society of America Bonus*

Problem 10. *2021 RMPh Problem 3: Water World*

Problem 11. *2019 GPhO T1 B: Glass of Water in Weightlessness*

Problem 12. *2011 Open EPhO (P182): Surface Tension*

Problem 13. *2009 NBPhO Problem 7: Soap Film*

4 Practice Problems

Problem 1. A thin soapy film is stretched in a square frame that has a length a. A small thin hair with negligible mass is fastened at two diagonal points in the frame. One part of the thread is pierced with a small needle so that the thread is strained with a tension force T. What is the length ℓ of the thread if it is known that $\ell > \sqrt{2}a$?

Problem 2. A soap film is created in a loop formed by a rectangular wire, and an inextensible light thread of length ℓ pulls the thread into a semicircle. Calculate the force F applied at the midpoint of thread AB such that the two parts of the thread turn into smaller semicircles as in the figure below.

Problem 3. A small coin has a thickness t, radius r, and density ρ_c. It is placed in a fluid of density ρ_f. What is the surface tension γ of the coin if its top surface is the same as the height of the liquid far away from the coin?

Problem 4. (2015 Online Physics Brawl) Verča likes to wash the dishes. When she's done, she takes a little boat and places it on the surface of water mixed with detergent. The boat is a small wooden board with thickness b, length ℓ and width w. Its length is parallel to the axis x. The surface tension in the x-direction is given by $(x) = \sigma_0 + xs$ and in the y-direction by $\sigma(y) = \sigma_0$, where $\sigma_0 = \text{const}$ and s is the gradient of surface tension. Determine the initial acceleration of boat after being placed on the water surface. The density of wood is ρ. If the acceleration is in the direction of increasing surface tension, it's positive; if it's in the opposite direction, it's negative. The contact angle of wood and water is β . The inclination of the boat is negligible

Problem 5. (2014 Online Physics Brawl) As it known, propagation of surface waves on a lake is not significantly influenced by gravity in the short-wavelength limit. Nevertheless there is a strong effect of surface tension which needs to be considered. Using dimensional analysis, determine the angular frequency of surface waves as a function of surface tension σ, water density ρ and the wave number $k = 2\pi/\lambda$, where λ is the wavelength. Assume that the dimensionless coefficient is equal to one in this case. Also, derive an expression for the group velocity.

Problem 6. (2018 Online Physics Brawl) Vítek was investigating some fluid properties using a water surface with surface tension σ_1. For this purpose, he made a thin cork board with mass m and side lengths b and c, which was – to his surprise – floating on the surface. He decided to get the board moving, so he added a bit of detergent to one of its shorter sides. The surface tension of the resulting soap (detergent) solution is σ_2. Help Vítek find the acceleration of the cork board. For simplicity, assume that water resistance is negligible and that the contact angles are always 90°.

Problem 7. (Thompson) The latent heat of vaporization of water is $H = 2.25 \times 10^9\ \text{Jm}^{-3}$. The surface tension of water is $\gamma = 72\ \text{mNm}^{-1}$. From these two facts, together with a simple theory, estimate the mean distance between neighbouring water molecules. (Hint: introduce the bond energy E between two molecules which you can assume to be constant.)

Problem 8. A mercury drop of density ρ_H and surface tension γ_H and a water droplet coated in Talcum powder are similar in shape and size. For a particular shape of a drop, the ratio of surface energy and gravitational potential energy bear a definite ratio regardless of any other parameter of the drop. What is the ratio of masses of the mercury drop to the water drop?

Problem 9. (RuPhO) Consider a square plate of sidelength a and thickness h on the water surface. The plate has a density ρ, water has a density ρ_w and a surface tension γ. What is the weight that the plate can support?

Problem 10. (2016 Rudolf Ortvay Competition) A droplet of mercury with radius R floats in zero gravity. If the droplet is placed into a weak homogeneous electric field E_0, it will slightly elongate in the direction of the field lines. Determine the equilibrium shape of the droplet! The surface tension of mercury is γ, and let us assume that $\varepsilon_0 E^2 R \ll \gamma$.

Problem 11. (2016 Seagull Competition) At the International Space Station, a drop of water with a radius of $R = 4$ cm floats in weightlessness. The drop is given a small angular acceleration, which is why it is spinning faster and faster. At what critical angular velocity ω does the droplet decompose into two pieces? The surface tension of water is $\gamma = 0.073$ N/m, water density $\rho = 1000$ kg/m^3.

Problem 12. (2016 Online Physics Brawl) Into how many smaller droplets could a larger water droplet break, if it fell from a h height? Consider all droplets to be spherical. The radius of the original droplet is r (with its center of mass at height h) and the droplets it breaks into should have equal sizes. The surface tension of water is σ and its density is ρ. We're only looking for an estimate of an upper bound of the number of droplets based on the energetic balance at the beginning and at the end.

Problem 13. Consider a water droplet as shown below. When it falls on a water hydrophobic surface, a smaller ring of smaller droplets assemble around the water droplet after the collision. Explain why this happens and under what conditions/parameters could we find and measure the dimensions of such a ring?

Problem 14. (Vietnam TST) A thin rectangular metal plate, one side of length a and the other which is very long relative to a is placed on the surface of a liquid that is completely wetted with respect to the metal. The glass plate is slowly raised to the highest position so that the liquid is still attached to the plate. Let φ be the angle formed by the tangent at any point M on the liquid boundary with the horizontal (Figure 2.1). Given that the barometric pressure is P_0, the density of the liquid is ρ, the surface tension of the liquid is σ, and the acceleration due to gravity is g

1. Find the x, y coordinates of M in the given $O(x, y)$ coordinate system as shown in the figure

below.

2. Given $a > 1.1\sqrt{\frac{\sigma}{\rho g}}$, Determine:

   (a) The maximum height h of the metal plate above the horizontal liquid surface

   (b) The minimum width b of the column of liquid adhering to the metal plate

   (c) The vertical force per unit length on the long side of the sheet given the unit length (long side) weight of the metal is P.

3. Redo the problem in the case $a < \sqrt{\frac{\sigma}{\rho g}}$.

- The radius of curvature R at a point on the curve can be determined by the formula $R = \left|\frac{\mathrm{d}s}{\mathrm{d}\varphi}\right|$.
- $\int \frac{\mathrm{d}\alpha}{\cos\alpha} = \ln\left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| + C$.

Problem 15. (Indonesia TST) A thin spherical liquid bubble forms in a small hole in the wall of a box containing a monatomic ideal gas ($C_V = \frac{3}{2}nR$). The size of the hole is so small that it is negligible. There is a piston that can be moved to change the volume of the room. The total gas in the chamber and bubbles is n moles. It is known that the initial total volume of the gas is $V_0 = 150\pi r_0^3$ (total box and bubbles) and the pressure P_0 is equal to atmospheric pressure. The cross-sectional area of the box is $S = 16\pi r_0^2$ where r_0 is the initial radius of the bubble. The surface tension of the bubble satisfies

$$\sigma = \sigma_0\left(\frac{r}{r_0} - 1\right).$$

with $\sigma_0 = r_0 P_0/30$. Notice that the bubble has two surfaces, namely an inner surface and an outer surface. The heat capacity of the bubble at a fixed radius is $C_r = \left(\frac{\partial Q}{\partial T}\right)_r = \frac{nR}{2}$. The initial temperature of the bubble is equal to the temperature of the gas. Assume that no heat can enter and leave the system.

1. At first the piston is compressed slowly enough to be considered quasi-static, but fast enough so that no heat from the gas moves to the bubble or vice versa. Determine how far the piston moves x if the bubble radius becomes $2r_0$. Also determine the gas temperature T_1 at that time.

2. If the internal energy of the bubble is U_b, determine the change in the bubble's internal energy with respect to the radius at constant temperature $\left(\frac{\partial U_b}{\partial r}\right)_T$

3. Now consider the case where the piston is held in place. After a long time , there is a transfer of heat between the gas and the bubbles so that the temperature is the same. Determine the final bubble radius and bubble temperature.

5 Appendix

5.1 Relationship of Surface Tension and Temperature

We can relate surface tension to the several state variables[1]. The first law of thermodynamics for a bubble surface yields

$$\mathrm{d}U = \gamma \mathrm{d}A + T\mathrm{d}S. \tag{11}$$

Furthermore, the differential form of Hemoltz's function shows

$$\mathrm{d}F = \gamma \mathrm{d}A - S\mathrm{d}T. \tag{12}$$

Therefore, we have a Maxwell relationship of

$$\left(\frac{\partial S}{\partial A}\right)_T = -\left(\frac{\partial \gamma}{\partial T}\right)_A. \tag{13}$$

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